13. Variedades

PGF 5005 - Mecânica Clássica

web.if.usp.br/controle

(Referências principais: Reichl, *The Transition to Chaos*,1992; Lichtenberg e Lieberman, *Regular and Chaotic Motion*, 1992)

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2023

Variedades

Conjunto de pontos associados aos pontos hiperbólicos.

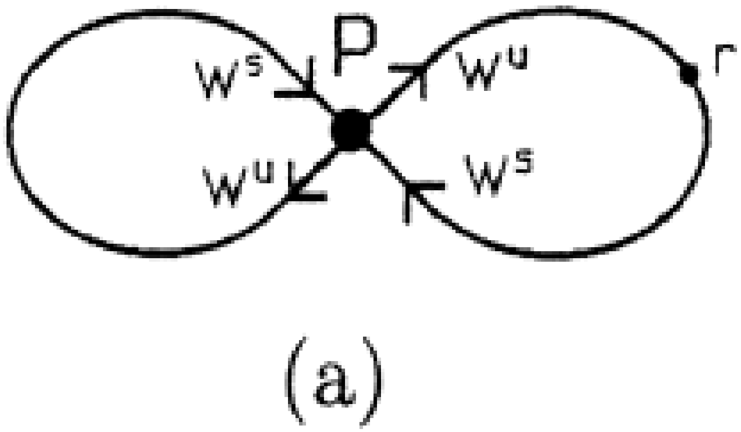
Em sistemas quase integráveis, substitutem as separatrizes.

Influenciam as órbitas ao seu redor e determinam a dinâmica nessa região.

Associadas à origem do caos.

For a nonintegrable system, a totally different behavior occurs. As one begins to perturb the map, the stable and unstable manifolds, $W^{(s)}$ and $W^{(u)}$, respectively, begin to oscillate and intersect one another transversally at an infinite number of places (see Fig. 3.2.6). For the case in Fig. 3.2.6.a, where $W^{(s)}$ and $W^{(u)}$ belong to the same hyperbolic fixed point, P, the points of intersection, r, r', r'', etc., are called homoclinic points, while for the case in Fig. 3.2.6.b, where $W^{(s)}$ and $W^{(u)}$ attach to separate hyperbolic fixed points, P and Q, the points of intersection, r, r', r'', etc., are called heteroclinic points. Homoclinic points in Fig. 3.2.6.a are mapped toward P by T_ϵ and T_ϵ^{-1}, but in opposite directions, while heteroclinic points in Fig. 3.2.6.b are mapped toward Q by T_ϵ and toward P by T_ϵ^{-1}. The homoclinic or heteroclinic points become more and more closely spaced as one approaches the hyperbolic fixed points and therefore, since area must be preserved by the map, the oscillations must grow in amplitude as one approaches the hyperbolic fixed points. Thus, near the hyperbolic fixed points, the motion becomes indescribably complicated. In fact, it has been shown that it is possible to embed a Bernoulli shift with an alphabet containing an infinite number of "letters" (the baker's map has two letters) in the neighborhood of each homoclinic or heteroclinic point.

Separatrizes
Sistemas integráveis



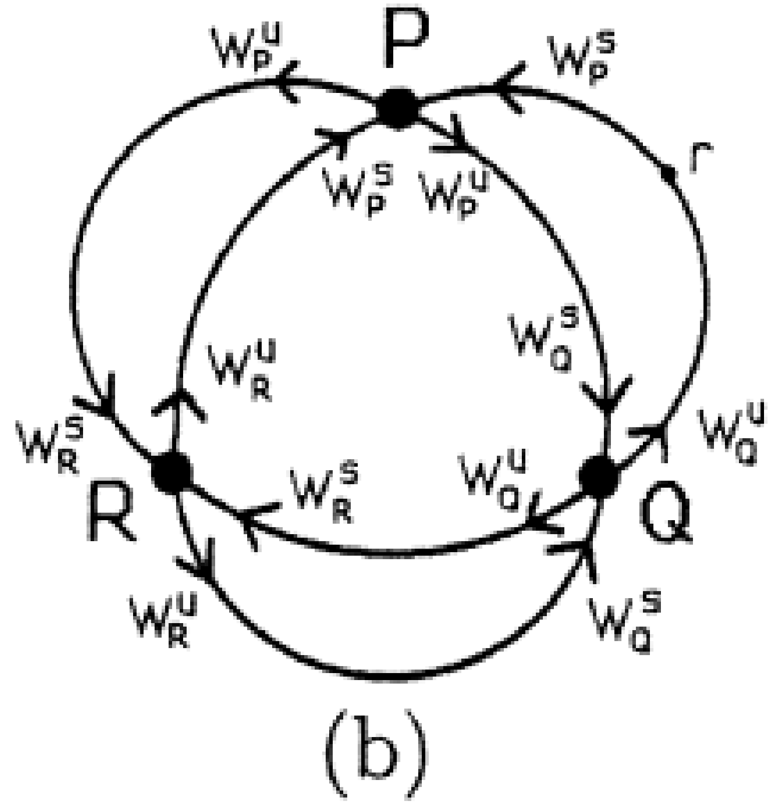


Figure 3.2.5. The stable and unstable manifolds, W^s and W^u, of hyperbolic fixed points for integrable systems join smoothly. (a) A point r on $W^{(s)}$ and $W^{(u)}$ is mapped to the same fixed point, P, by T_o and T_o^{-1}. (b) A point r on $W^{(s)}$ and $W^{(u)}$ is mapped to fixed point P by T_o and to fixed point Q by T_o^{-1}.

Sistemas Quase Integráveis

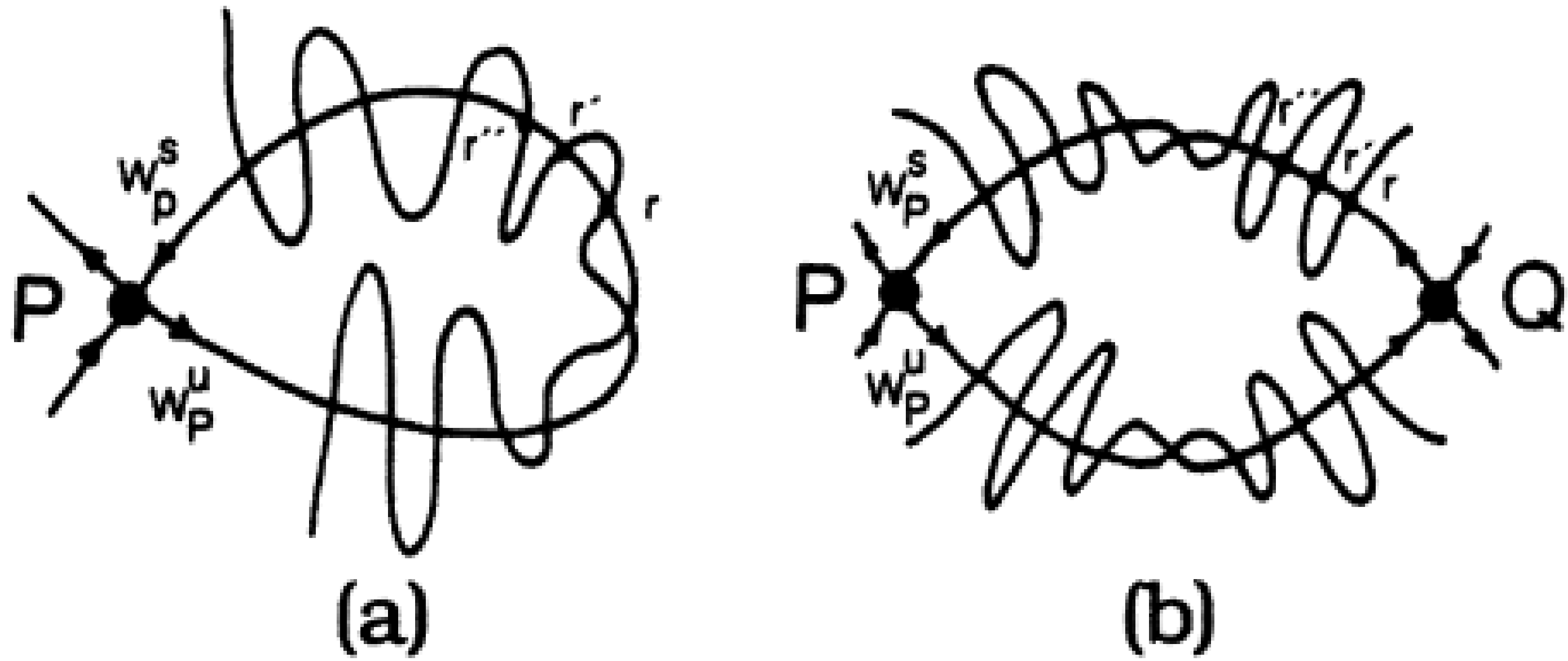


Figure 3.2.6. For nonintegrable systems, the stable and unstable manifolds no longer join smoothly but oscillate and intersect transversally. (a) For stable and unstable manifolds that approach the same fixed point, the intersections, r, r', r'', etc., are called homoclinic points. (b) For manifolds that approach different hyperbolic fixed points, the points of transversal intersection, r, r', r'', etc., are called heteroclinic points.

Mapa de Duffing
(ligeiramente dissipativo)

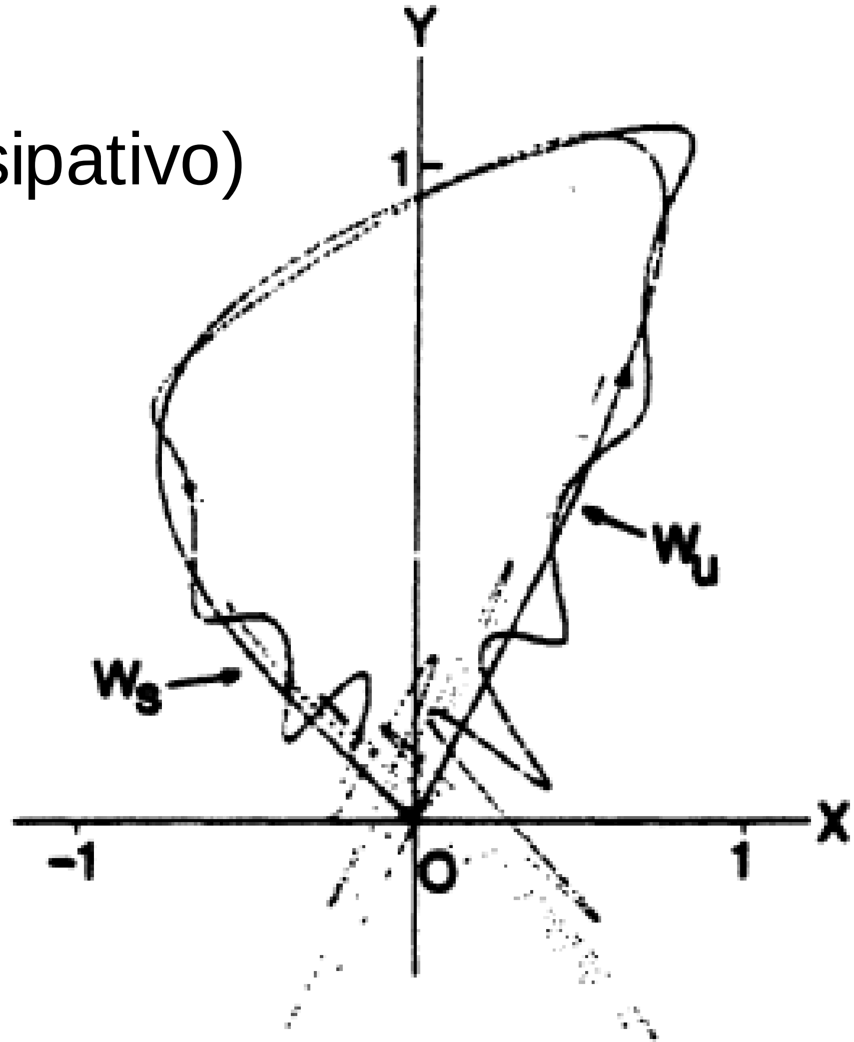
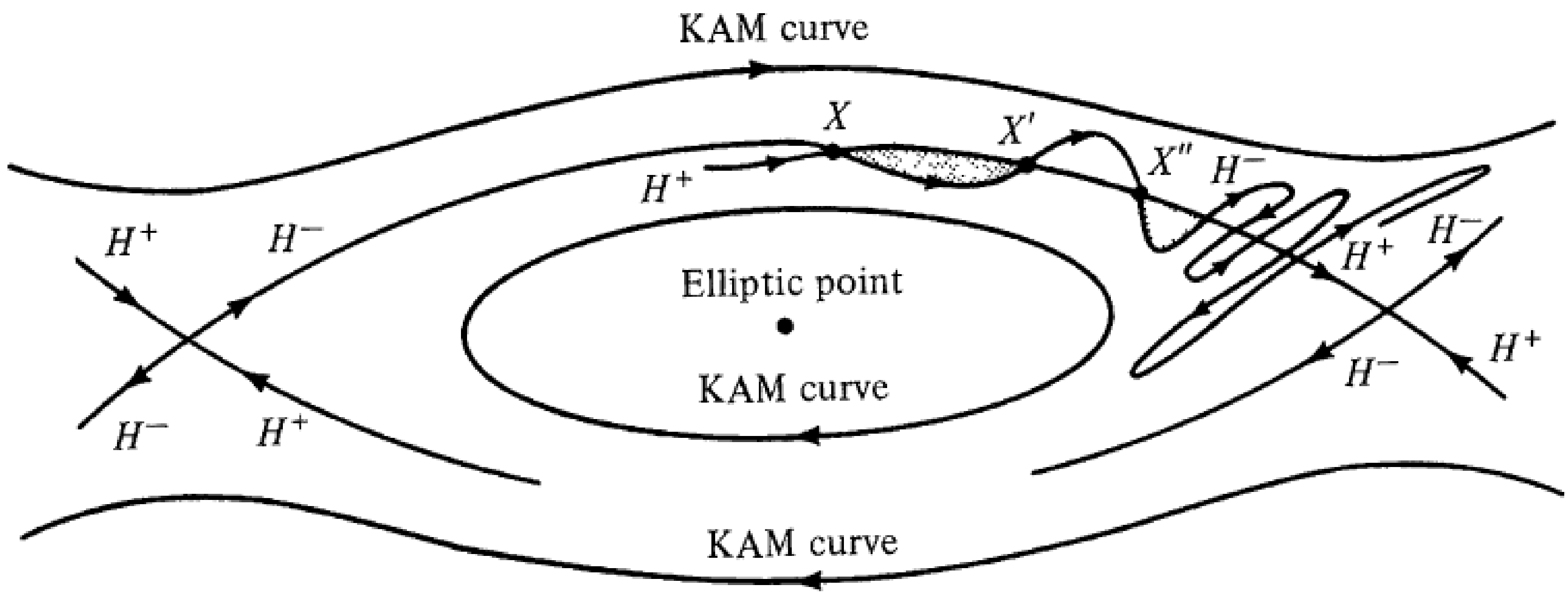
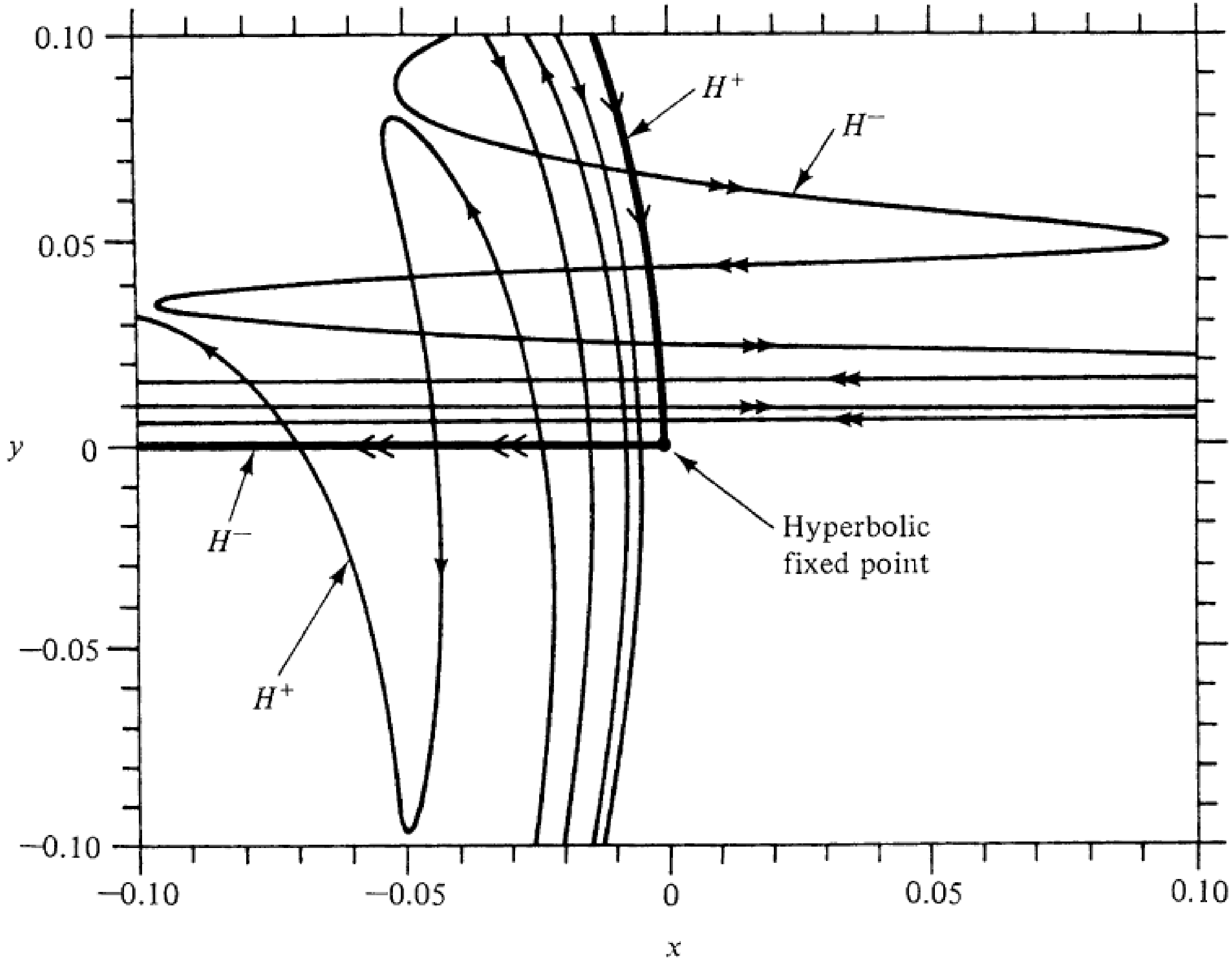


Figure 3.2.8. The unstable manifold, W^u, and stable manifold, W^s, manifolds computed numerically for the Duffing map, $x_{n+1} = y_n$, and $y_{n+1} = qy_n - y_n^3 - Jx_n$, where a and J are constants [Yamaguchi and Mishima 1984]. Note that this mapping is slightly dissipative but still gives a very clear picture of the behavior of the stable and unstable eigencurves.



KAM curve
X
X'
X''
H+
H−
H+
H−
H−
H+
Elliptic point
H+
H−
H+
H−
KAM curve
H−
H+
KAM curve

(a)



(b)

Figure 3.4. Illustrating the effect of a homoclinic point on the generation of stochasticity near a separatrix. (a) The stable (H^+) and unstable (H^-) branches of the separatrix intersect infinitely many times. (b) Detail of the intersections near the hyperbolic fixed points (after Dragt and Finn, 1976a).

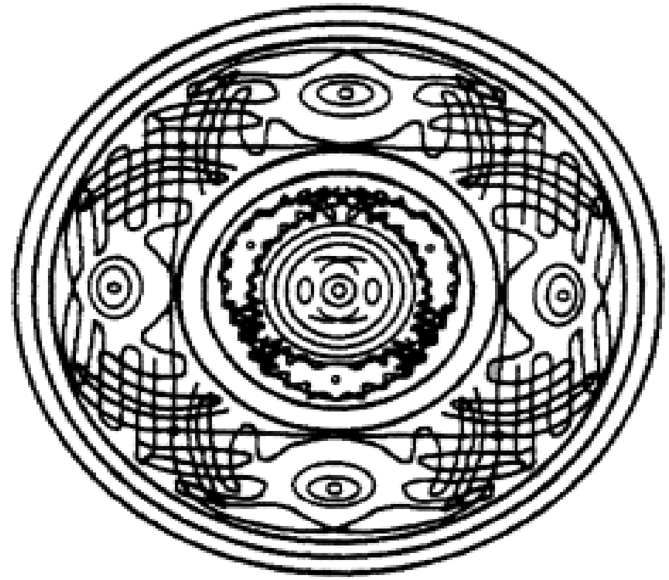


Figure 3.2.9. The twist map of a nonintegrable system contains a complex mixture of chaotic and regular trajectories.

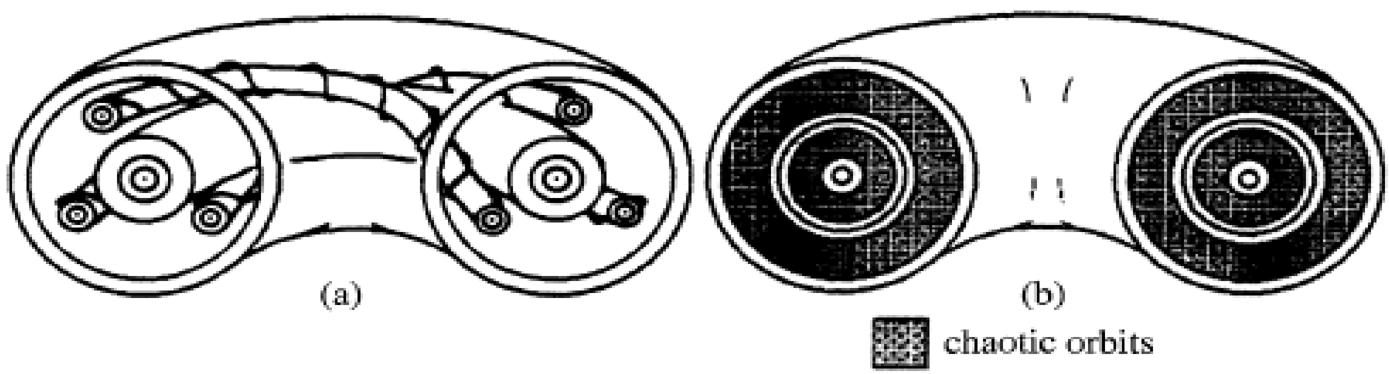
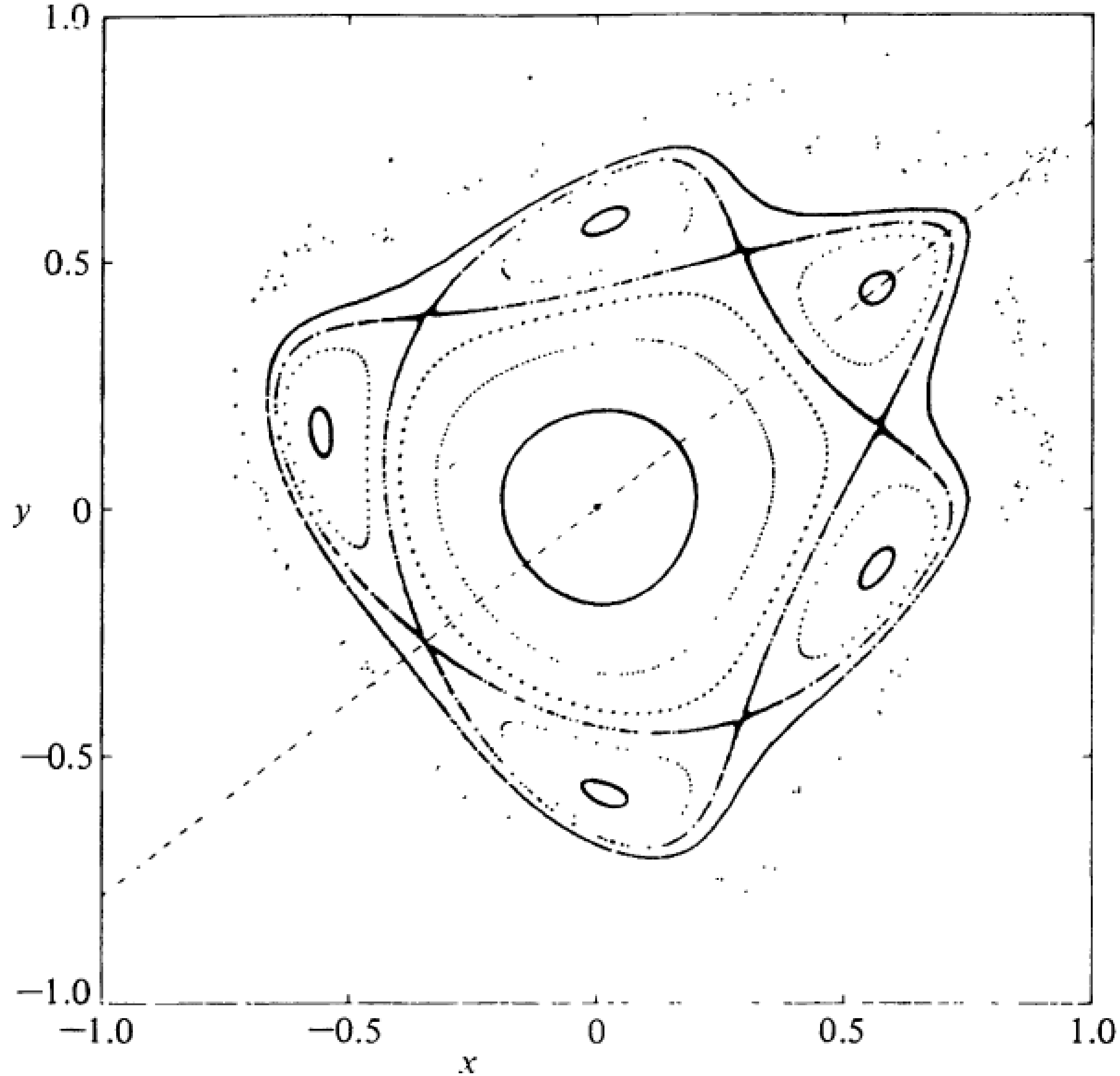


Figure 3.2.10. (a) The torus is composed of a complex tangle of smaller tori. (b) Layers of chaotic trajectories are sandwiched between regular tori.

Exemplo Numérico

Here we illustrate some of the phenomena previously discussed in this section with two examples, the first being the quadratic twist mapping studied by Hénon (1969):

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{bmatrix} x_0 \cos\psi - (y_0 - x_0^2)\sin\psi \\ x_0 \sin\psi + (y_0 - x_0^2)\cos\psi \end{bmatrix}, \qquad (3.2.40)$$



(a)

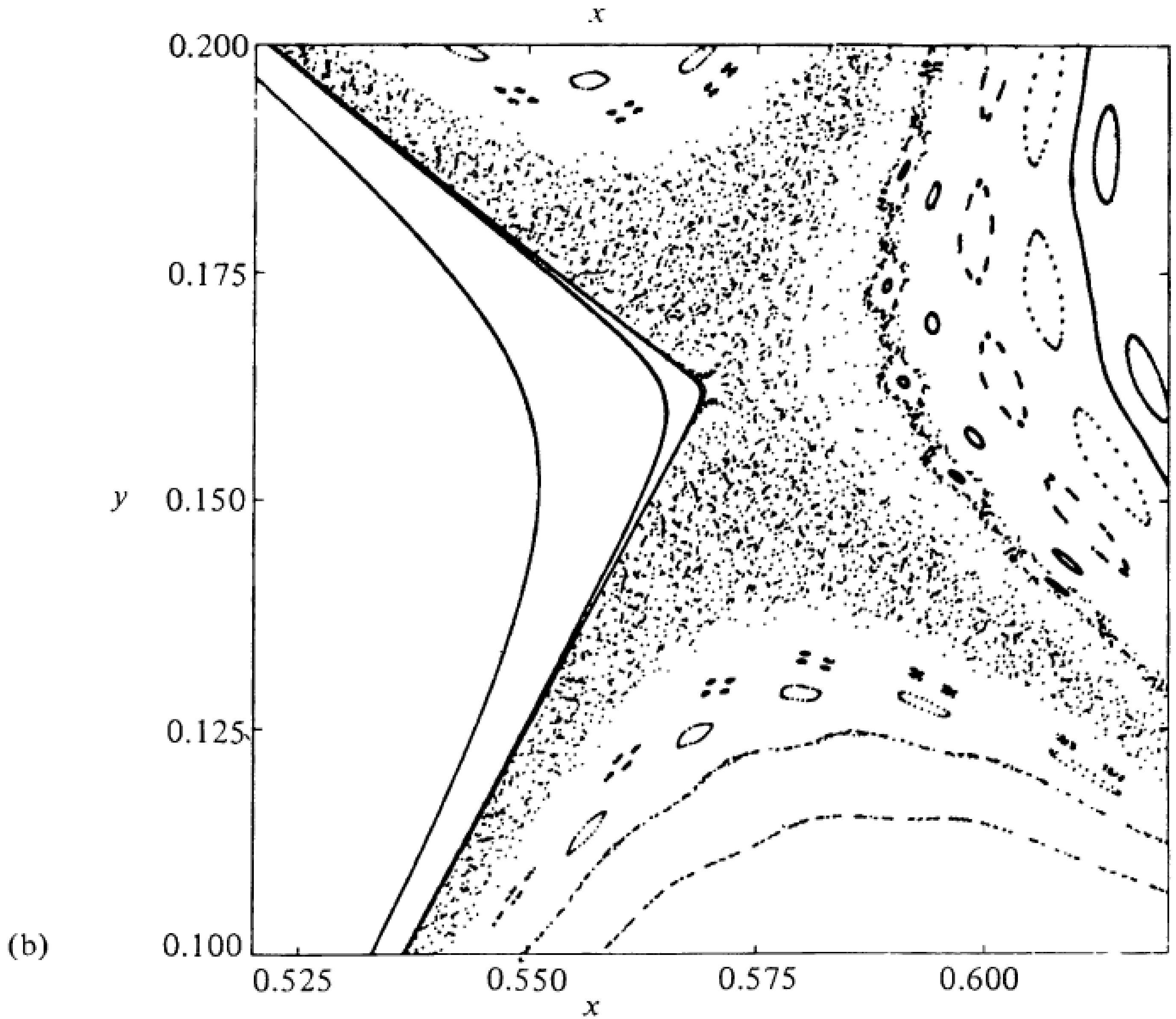


Figure 3.6. Trajectories of the Hénon mapping [Eqs. (3.2.40)] with $\alpha = 0.2114$. (a) Mapping including the origin and the first island chain; (b) expanded mapping near a separatrix of the first island chain. Each island chain, etc., is generated from a single (x_0, y_0) (after Hénon, 1969).

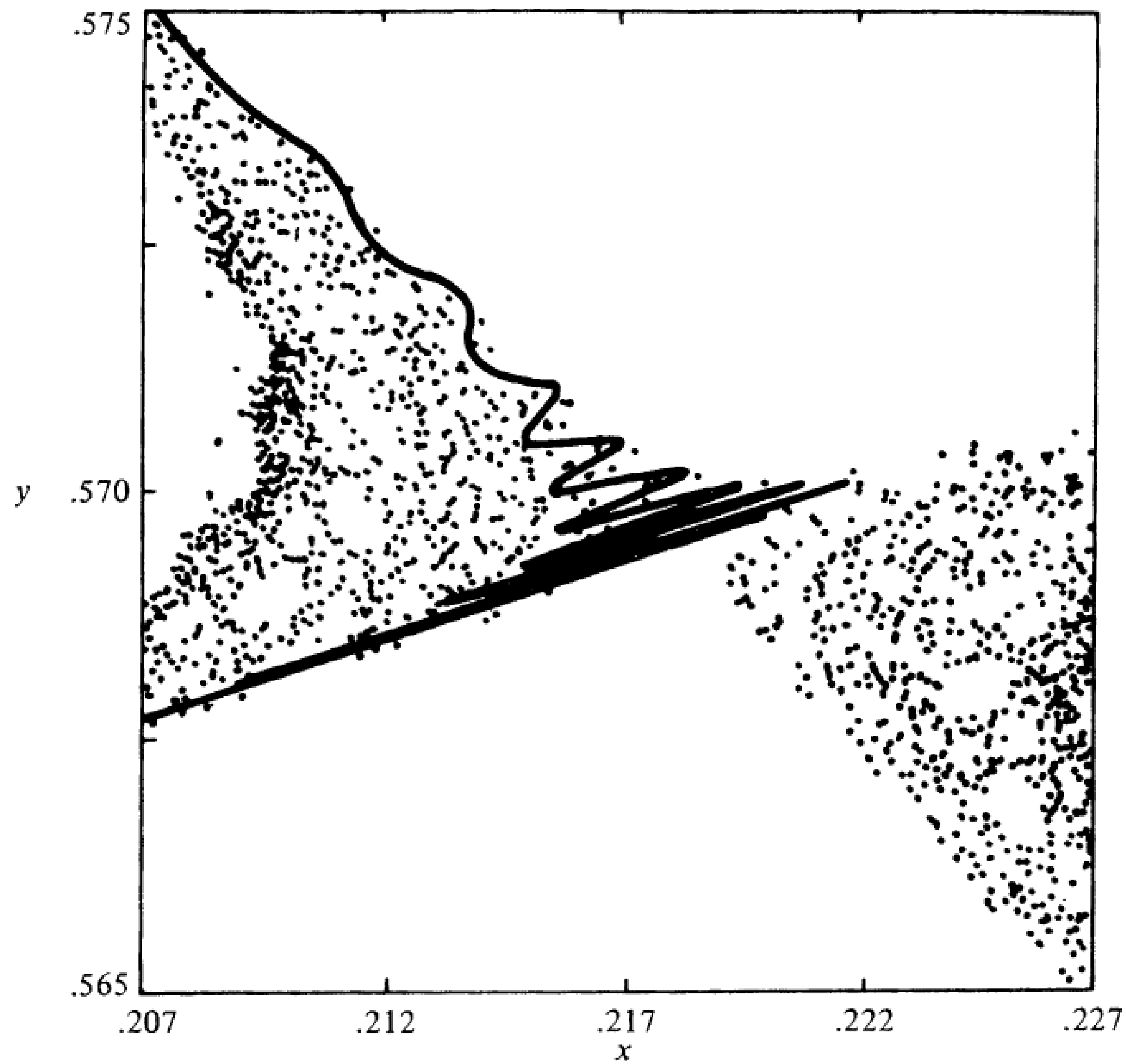


Figure 3.7. Transformation of a small line segment of initial conditions on the separatrix (solid curve) through sufficient mapping iterations to develop the behavior of the separatrix mapping near a homoclinic point (after Berry, 1978).

Efficient manifolds tracing for planar maps

Based on the article: D. Ciro, I. L. Caldas, R. L. Viana and T. E. Evans, Chaos **28**, 093106 (2018).

Motivation

- Invariant manifolds are fundamental to understanding the underlying structure of the chaotic orbits of planar maps.

- Limitation on the manifold length obtained numerically: increase in length of the manifold segments due to the stretching and folding mechanism inherent to the chaotic dynamics.

Objectives

- Introduce a reliable and efficient numerical procedure to obtain the invariant manifolds of saddle points in maps and flows.

- Understand the features of chaotic double-well Hamiltonian systems in terms of the interaction between invariant manifolds of different saddles.

I - Invariant manifolds of planar maps

Let x* be a fixed saddle point of a diffeomorphism T: Σ→Σ, the stable (W_s) and unstable (W_u) manifolds are defined as

$$\begin{aligned}\mathcal{W}_u(x^*) &= \{x \in \Sigma : T^{-n}(x) \to x^*, \text{ as } n \to \infty\} \\ \mathcal{W}_s(x^*) &= \{x \in \Sigma : T^{n}(x) \to x^*, \text{ as } n \to \infty\}\end{aligned}$$

Saddles lead to sensitivity to initial conditions uder repeated application of a map T.

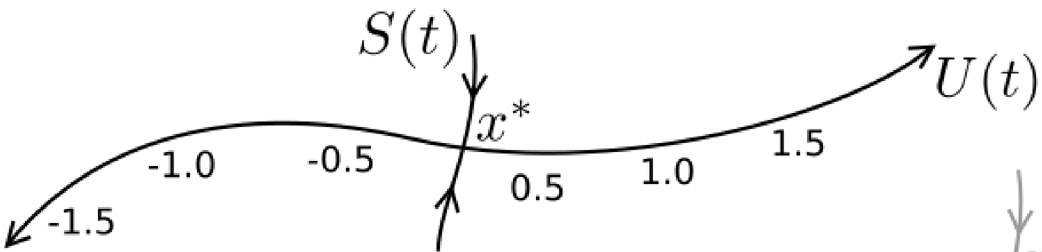
$$x_n = T^n(x_0)$$

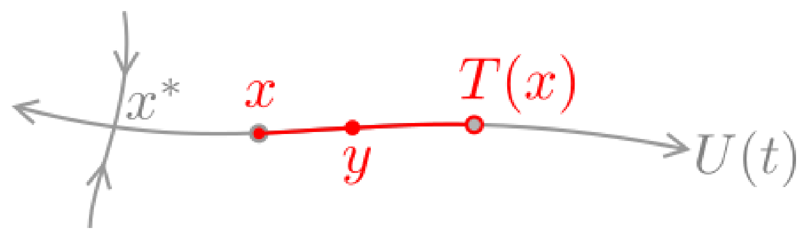
Decomposition in primary segments

Assume that it is possible to produce a parametric representation of the stable and unstable manifolds. For instance:

$$W_u(x^*) = \{x = U(t), -\infty < t < \infty \text{ and } x^* = U(0)\}$$

$$W_s(x^*) = \{x = S(t), -\infty < t < \infty \text{ and } x^* = S(0)\}$$

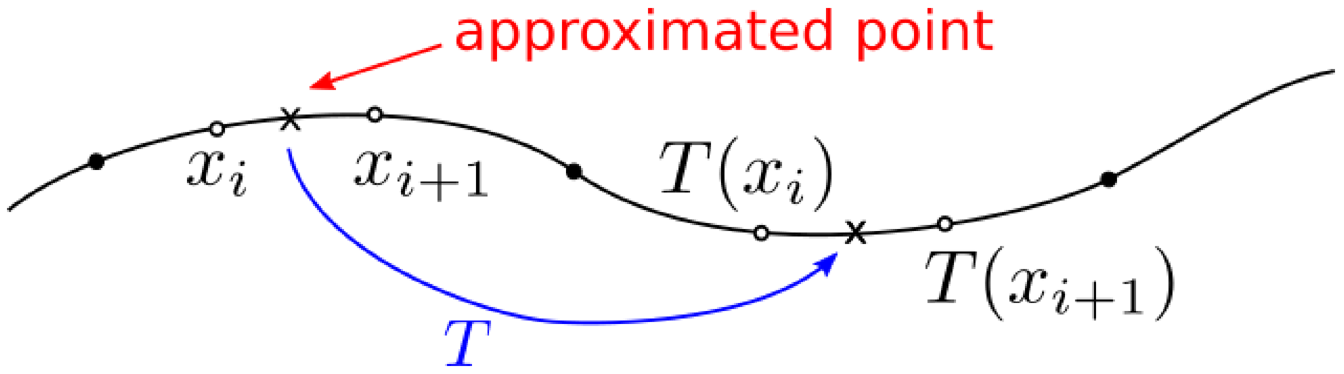




A primary segment of the unstable (stable) manifold based on x is a continuous set defined by:

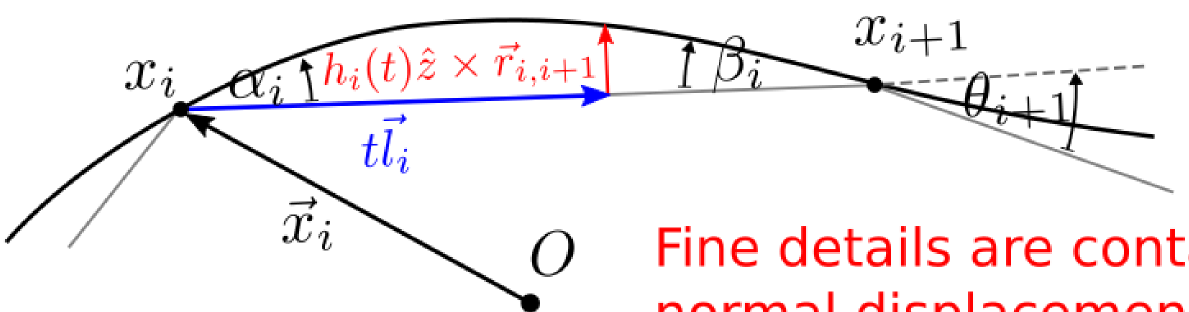
$$P_u(x) = \{y = U(t) | t_x \leq t < t_{T(x)}, \text{ where } U(t_x) = x\}$$

The approximated calculation method consists in generating the new points in the segment previous to the optimized one. This involves generating a parametric representation of the previous segment.



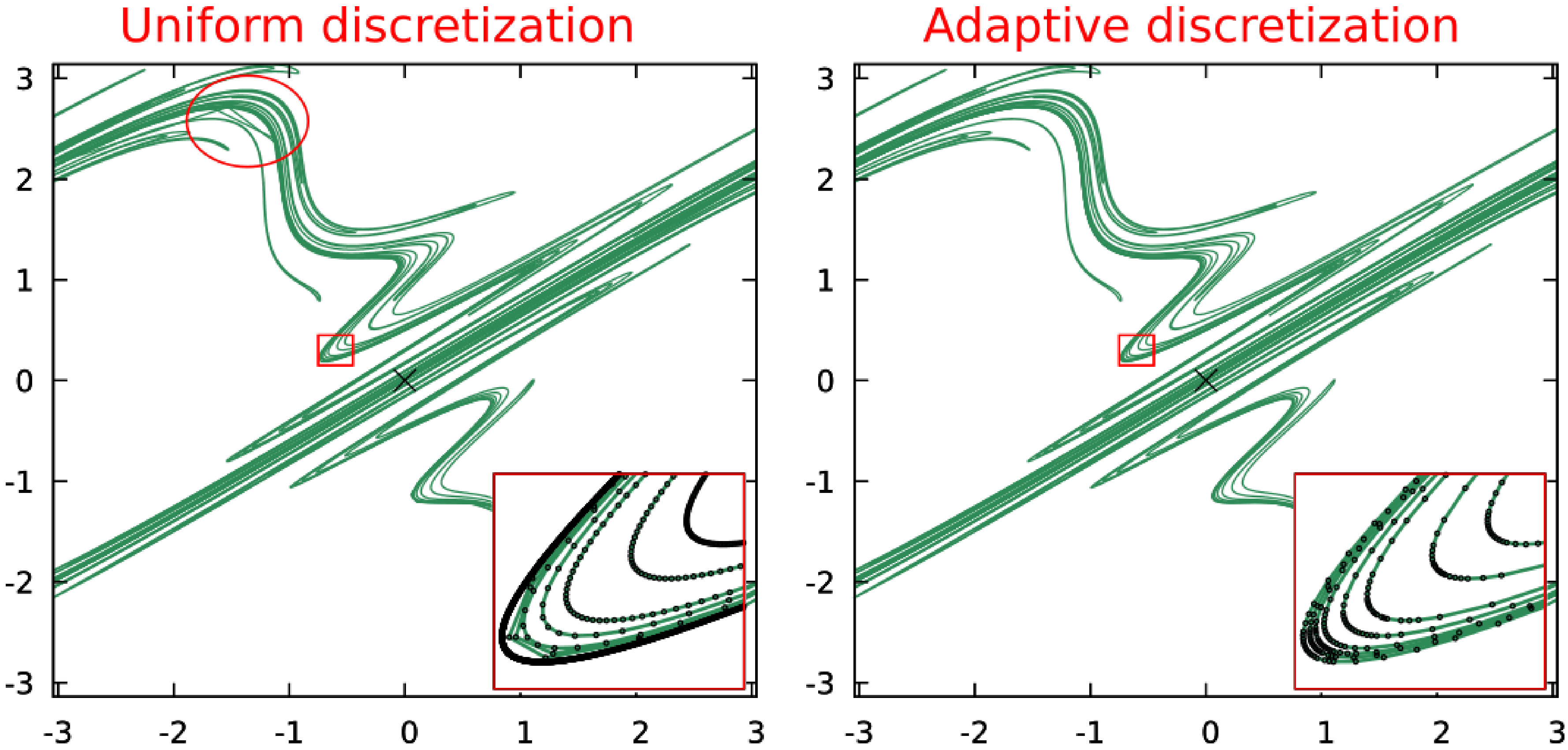
This requires interpolation based on the existing nodes

$$\vec{f}_i(\gamma_i, s) = \vec{x}_i + t\vec{l}_i + h_i(\gamma_i, t)\hat{z} \times \vec{l}_i$$



Fine details are contained in the normal displacement function.

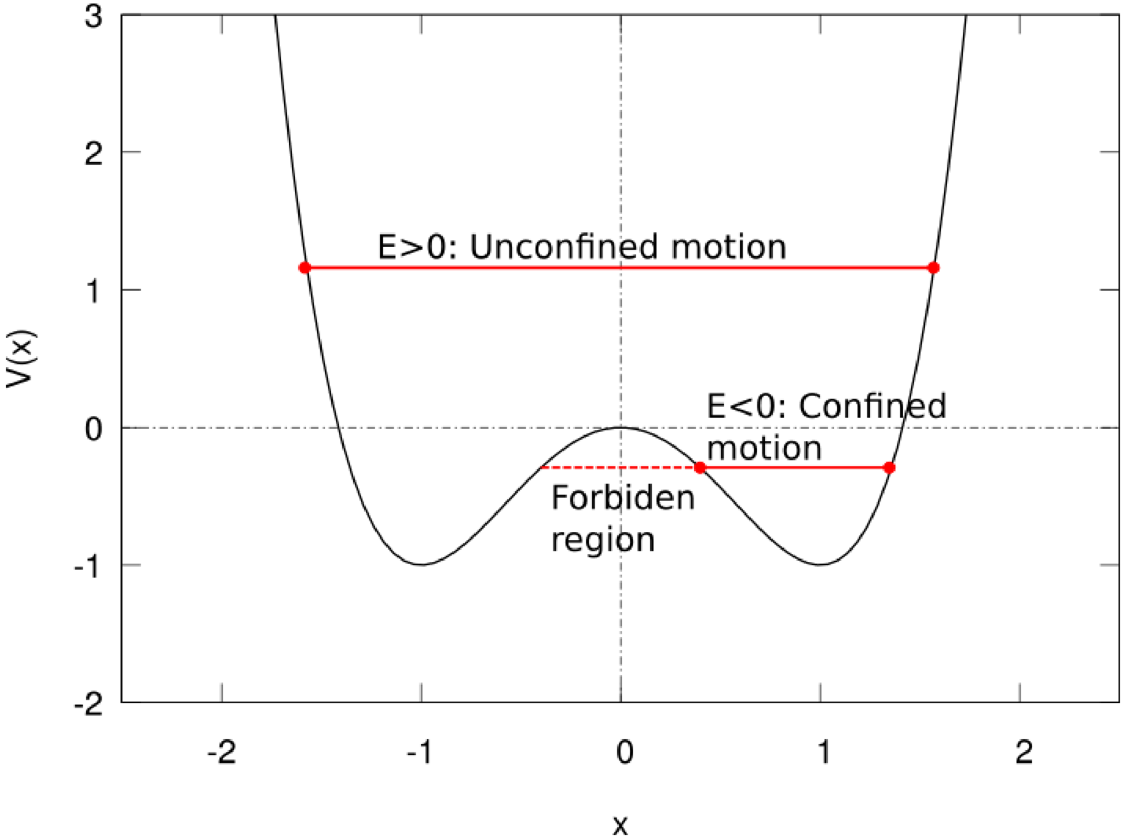
Example: The Chirikov-Taylor map k=1.5, central saddle.



With 25 primary segments the uniform discretization becomes insufficient (5000 points per segment).

II – Double-Well Systems

Consider the one-dimensional motion of a particle of mass m=2 in a double-well potential

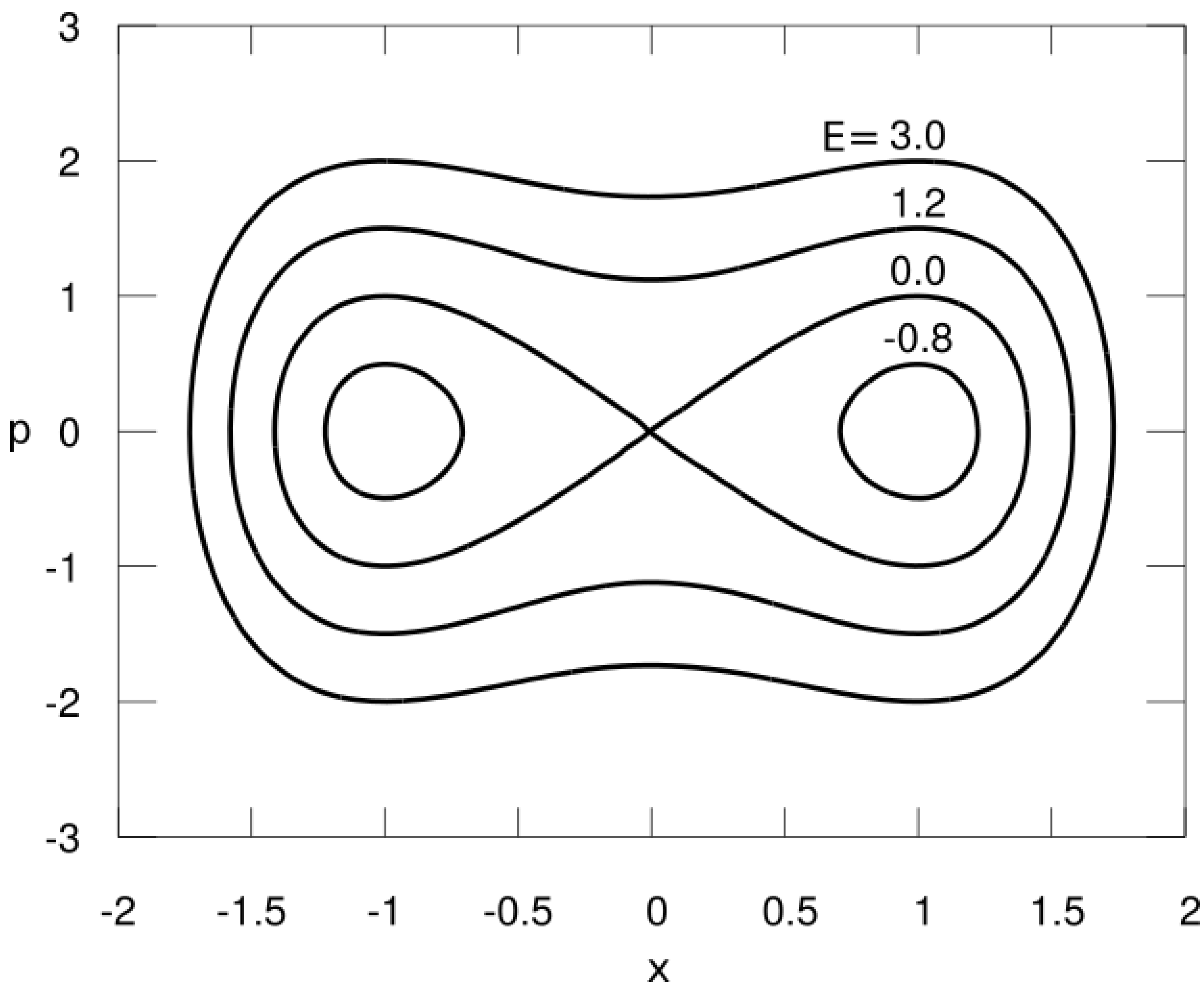


$$V(x) = -2x^2 + x^4$$

Which leads to the Hamiltonian function:

$$H(x, p) = \frac{1}{4}p^2 - 2x^2 + x^4$$

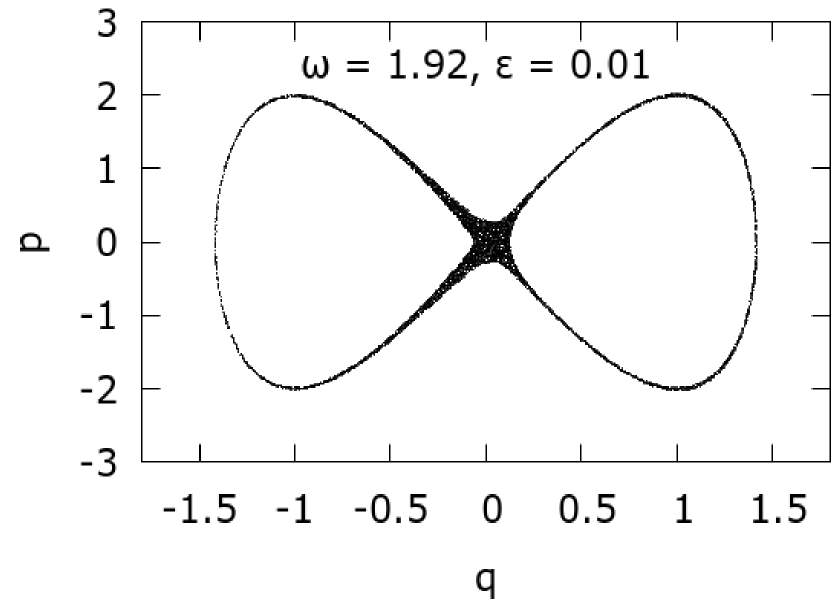
The resulting motion is completely integrable.

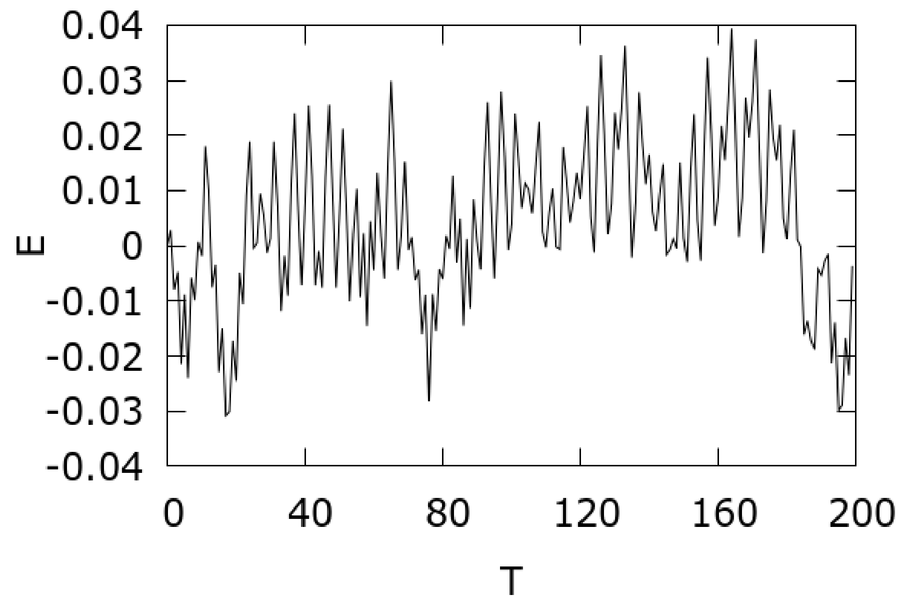


In the autonomous case, phase space trajectories move periodically along the level sets of the Hamiltonian function and the separatrix (E=0) separates trapped from untrapped motions.

In the non-autonomous case, the Hamiltonian function is perturbed periodically and the energy fluctuates.

Phase space trajectories near the separatrix are no longer confined or unconfined, but describe an irregular motion.

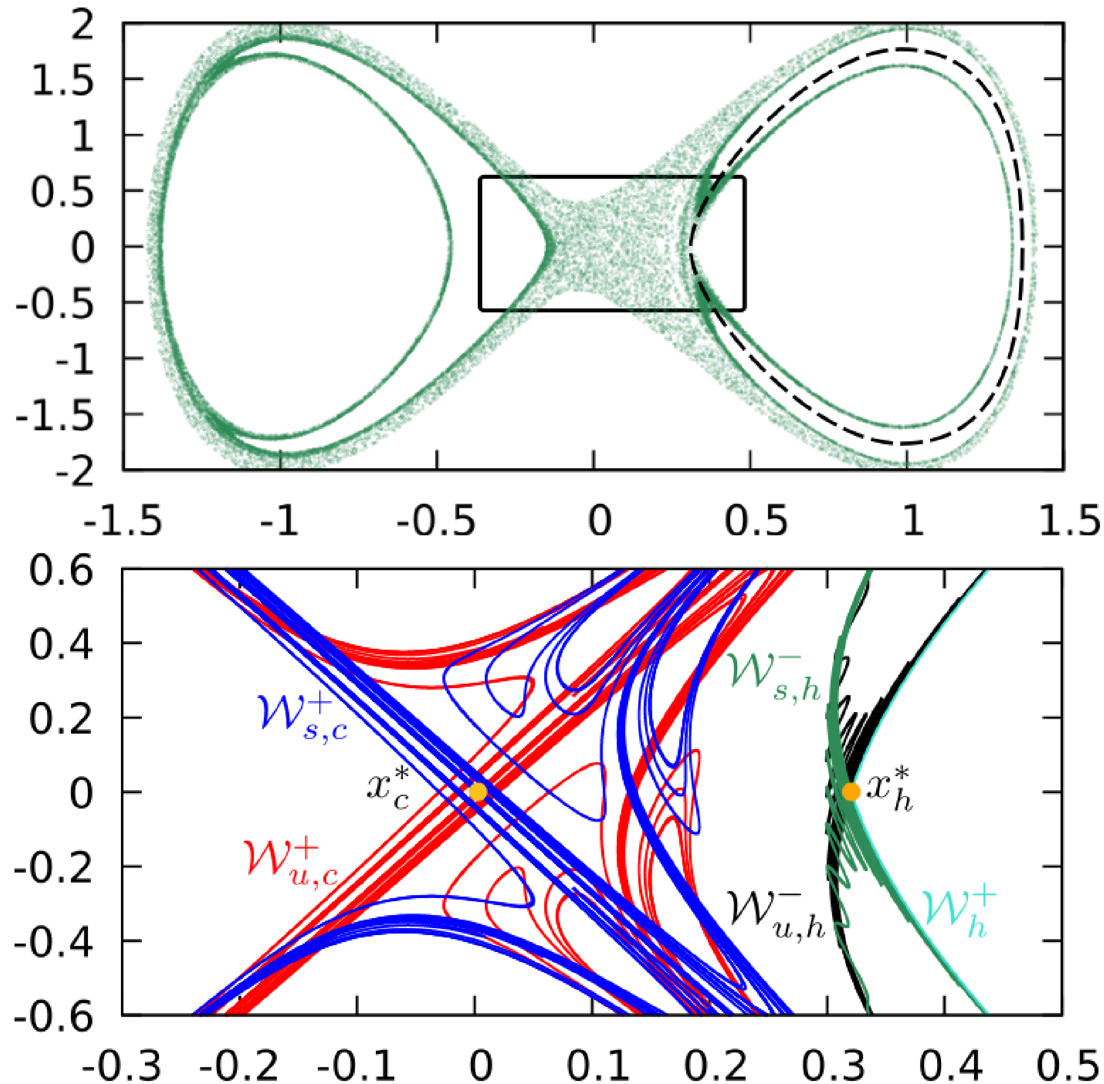




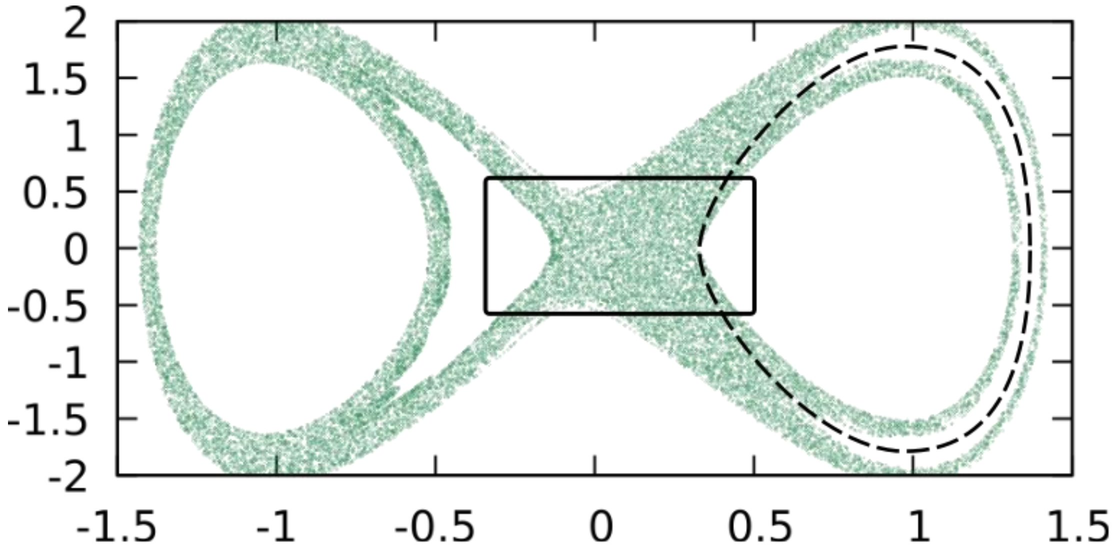
$$H(x,p) = \frac{1}{4}p^2 - 2x^2 + x^4 + \epsilon\cos(\omega t)$$

For ω = 1.5 and ε = 0,025 the chaotic region is wider and the orbits spend long time close to a period one ressonant island near the separatrix.

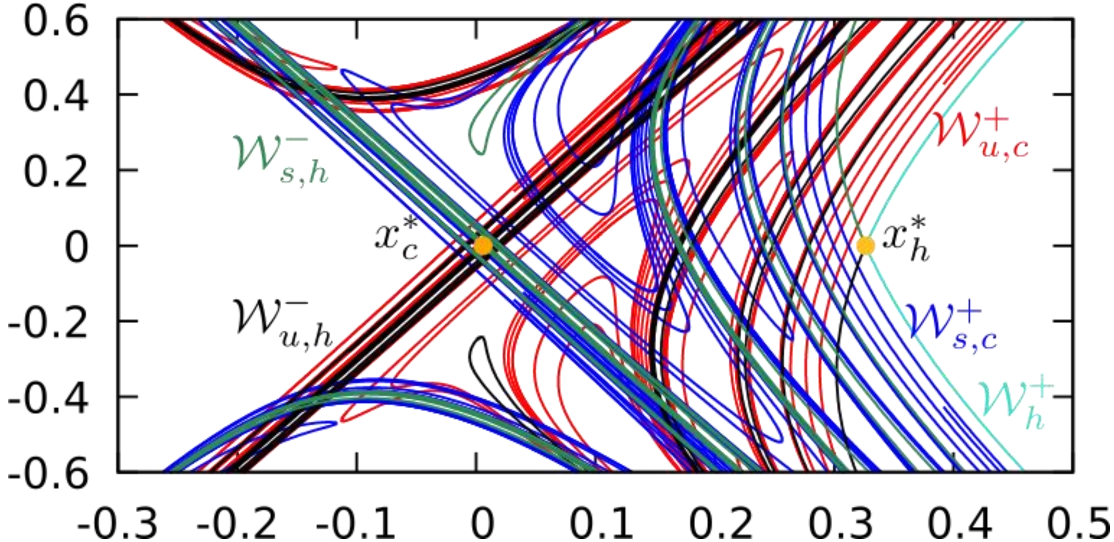
For a finite invariant manifolds tracing for the central saddle xc and the island helical saddle xh there is no observable interaction between the curves, and consequently orbits spend long times near one manifold before moving to the other.



For ω = 1.5 and ε = 0,05 the chaotic region becomes even wider and the orbits show no prefereable region in phase space.

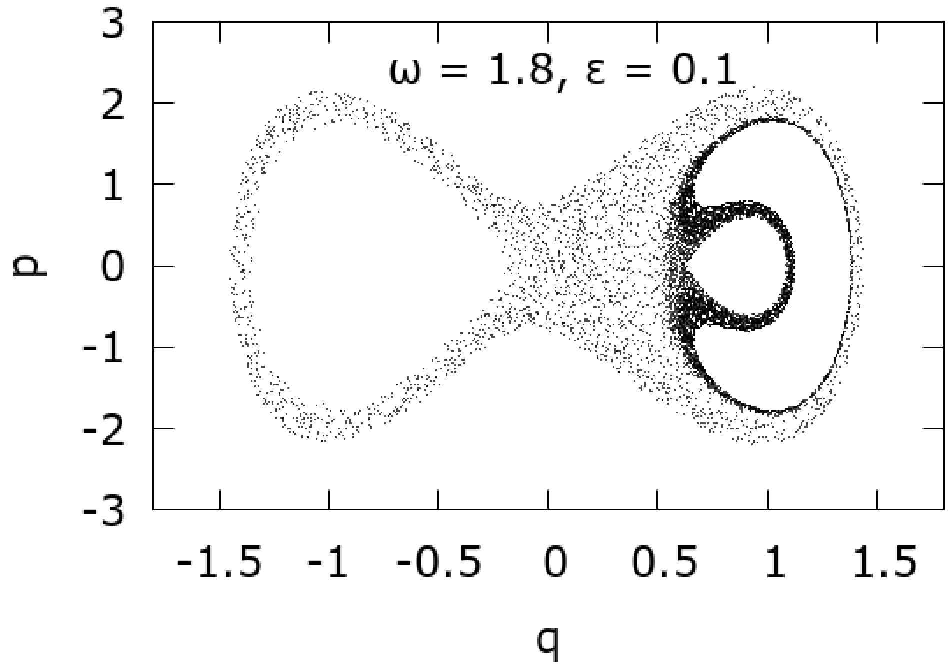


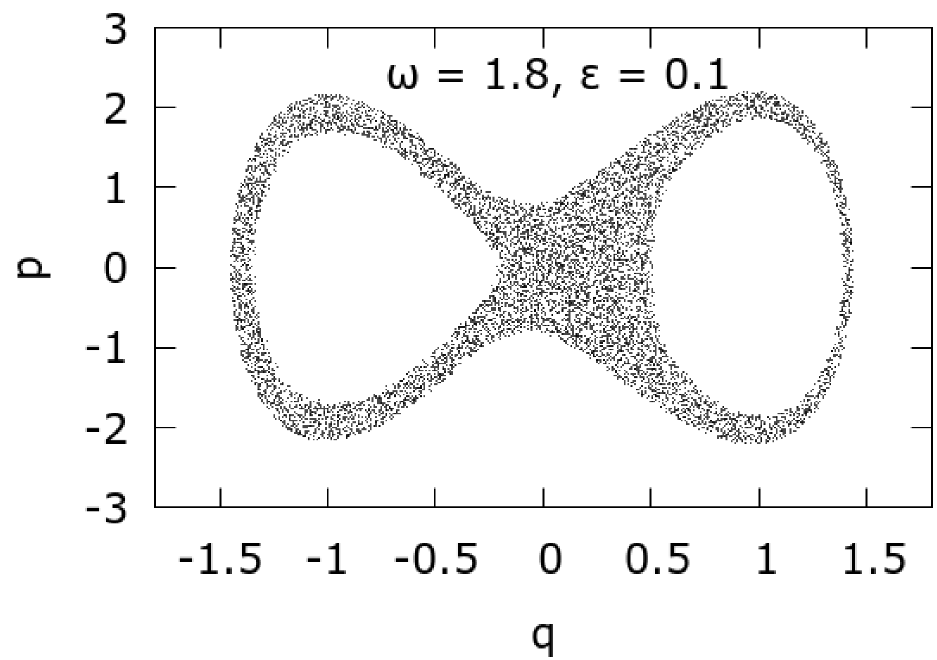
In this case, the finite invariant manifolds tracing reveals intricate heteroclinic crossing between the stable and unstable manifolds of different saddles, leading to an enhanced more homogeneous transport.

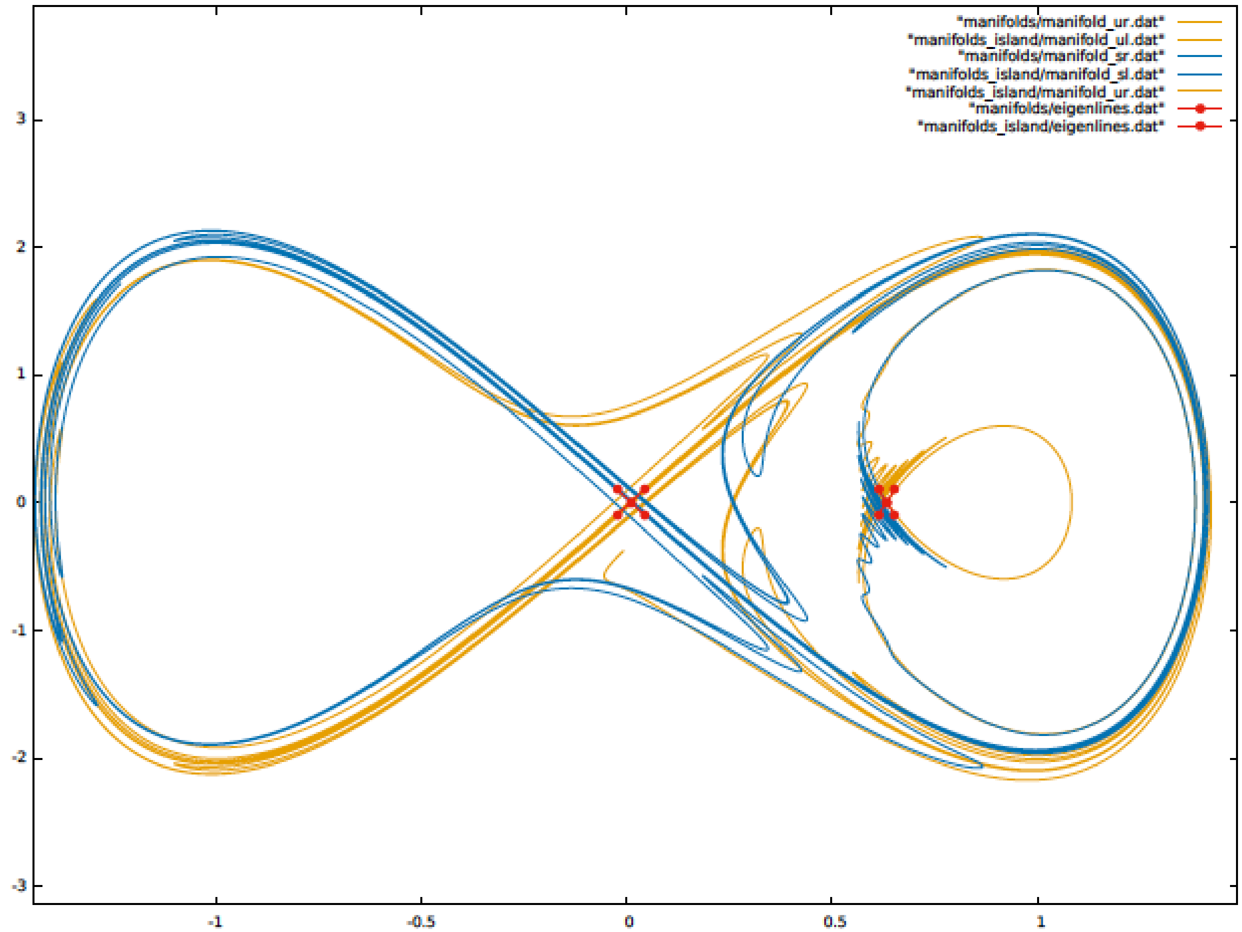


Different initial conditions may exhibit very different behaviors deppending on the manifolds influencing their motion.

In this case a particle with initial energy Eo=-0.57 (top) spend a long time near the ressonant island, while a particle with Eo=0.0 presents a more uniform distribution in phase space.







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"manifolds_island/manifold_ul.dat"
"manifolds/manifold_sr.dat"
"manifolds_island/manifold_sl.dat"
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3
2
1
0
-1
-2
-3
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1

Conclusions

We have introduced a new reliable and efficient procedure to trace invariant manifolds with high precision.

Applications included the understanding of chaotic transitions in continuous time bi-stable systems (e.g Duffing system), and the manifolds tracing for discrete symplectic maps (e.g. Chirikhov-Taylor Map and the conservative Hénon map).

A formal presentation of the method and its applications can be found in: D. Ciro, I. L. Caldas, R. L. Viana and T. Evans, Chaos 28, 093106 (2018).

Conclusions

- We have studied some general features of the chaotic dynamics of conservative open systems with the aid of a simple archetypal system, the area-preserving Hénon map.
- In order to explain different scenarios in dynamical trapping, we studied the changes in a set of homoclinic connections and another of heteroclinic connections. Both sets were obtained approximatelly by an adaptive refinement procedure.
- Transit times were associated with orbits around the homoclinic connections of the periodic saddles, while the unstable fixed points played a more structural role, delimiting the open chaotic region.